Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 53-2 (2020) 2921–2926

# EDC: Exact Dynamic Consensus

**Rodrigo Aldana-López** * **Rosario Aragüés** * **Carlos Sagüés** *

* *Departamento de Informatica e Ingenieria de Sistemas (DIIS) and Instituto de Investigacion en Ingenieria de Aragon (I3A), Universidad de Zaragoza, Zaragoza 50018, Spain. (e-mail: rodrigo.aldana.lopez@unizar.es, raragues@unizar.es, csagues@unizar.es)*

**Abstract:** This article addresses the problem of average consensus by a multi-agent system when the desired consensus quantity is a time varying signal, in particular the average of individual time varying signals localized at the agents. Although this problem has been addresses in existing literature by linear schemes, only bounded steady-state errors has been achieved. In this work, we propose a new exact dynamic consensus algorithm which leverages high order sliding modes to achieve zero steady-state error of the average of time varying reference signals in a group of agents. Moreover, our proposal is also able to achieve consensus to high order derivatives of the average signal, if desired. Finally, the effectiveness and advantages of our proposal are shown with concrete simulation scenarios.



*Keywords:* dynamic consensus, high order sliding modes, multi-agent systems

## 1. INTRODUCTION

In the context of cyber-physical systems, there are a lot of scenarios where the coordination of many subsystems is needed. There is no doubt that distributed solutions are preferred over centralized ones when, big networks of agents are involved in the scenario (Kia et al., 2019). This is so, since distributed solutions scale better with respect to the size and topology of the network, and are more robust against failures (Kar and Moura, 2008). Static consensus, where all subsystems (herein referred as agents) manage to agree on a static value such as the average of certain quantities of interest, is a widely studied topic, see for example (Olfati-Saber et al., 2007; Gómez-Gutiérrez et al., 2018). On the other hand, consensus towards a time-varying quantity has recently attracted attention due to its potential applications such as distributed formation control, distributed unconstrained convex optimization, distributed state estimation and distributed resource allocation, just to give some examples (see Kia et al. (2019)).

The most popular approach for dynamical consensus is to rely on a linear protocol similar to the one used in static consensus, but with the slight modification in which the error between the internal agent state and its local reference signal is shared to other agents instead of the state itself. This same approach can be reformulated as communicating only the internal state, with the drawback of using the derivative of the reference signal in the local evolution of the state of each agent (see Kia et al. (2019) for more details on this procedure). However, since this approach relies on a linear system stabilization, were the derivative of the average of the reference signals is not known globally, then, it can't be feed-forwarded to all agents in order to achieve zero steady-state error. Indeed, only practical stability towards consensus can be guaranteed, where the accuracy of the steady state depends on the bounds of the derivative of the reference signals, and its reduced as the connectivity is increased.

Zero steady state error has been achieved successfully by taking advantage of sliding mode control theory, where the steady state error is cancelled out by means of discontinuous protocols around consensus, see for example (Rahili and Ren, 2017). However, this approaches have the drawback that they induce chattering phenomenon to the system. Moreover, they need the derivative of the reference signals or even the reference signals themselves to be bounded, which can be restrictive in many situations.

### 1.1 Contributions

In this work, we propose a new Exact Dynamic Consensus (EDC) algorithm which leverages High Order Sliding Modes (HOSM) (Levant, 2008) to achieve zero steady-state error of the average of time varying reference signals of a group of agents. In this case, it is only required that a certain high order derivative of the reference signals is known to be bounded by a known constant. Moreover, this method successfully achieves consensus not only to the average of the reference is signals, but its derivatives. To the best of our knowledge HOSM hasn't been used in the context of dynamical consensus for this purpose.

This article is organized as follows. Section 2.1 draws some notation used through the article, Section 2.2 presents some concepts of algebraic graph theory needed to describe our result, Section 2.3 shows an important result of HOSM which is the basis of our proposal. Furthermore, Section 3 presents a formal statement for the problem being

* This work was supported by projects COMMANDIA SOE2/P1/F0638 (Interreg Sudoe Programme, ERDF), PGC2018-098719-B-I00 (MCIU/ AEI/ FEDER, UE) and DGA T45-17R (Gobierno de Aragon). The first author was funded by "Universidad de Zaragoza - Santander Universidades" program.


10.1016/j.ifacol.2020.12.966

addressed in this work. The main result is presented in Section 4 and its effectiveness and advantages are supported by a simulation example in Section 5. Finally, some conclusions are drawn in Section 6.

## 2. PRELIMINARIES

### 2.1 Notation

The function sign$(x)$ is defined as 1 for $x \geq 0$ and $-1$ for $x < 0$. Moreover, the notation $\lceil x \rfloor^{\alpha} = |x|^{\alpha}\text{sign}(x)$ will be used through this document. The symbols $\dot{x}(t), \ddot{x}(t)$ represent the first and second time derivatives of $x(t)$ whereas $x^{(\mu)}(t)$ for $\mu \geq 0$ represent the $\mu$-th time derivative of $x(t)$. Furthermore, $\mathbb{1} = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$, where the dimension $n$ is defined depending on the context. Italic indices $i, j$ will be used when referring to agents in a multi-agent system.

### 2.2 Graph theory

The communication network between agents can be modeled as an undirected graph. An undirected graph $\mathcal{X}$ consists of a node set $\mathcal{V}$ of $n$ nodes and an edge set $\mathcal{E}$ of $\ell$ edges. An edge from node $i$ to node $j$ is denoted as $(i, j)$, which means that node $i$ can communicate to node $j$ in a bidirectional way. If there is a path between any two vertices by means of edges of the graph $\mathcal{X}$, then $\mathcal{X}$ is said to be connected.

The adjacency matrix $A \in \mathbb{R}^{n\times n}$ of a graph has the property that its components $a_{ij} = 1$ when $(i, j)$ is an edge which is part of the graph and $a_{ij} = 0$ otherwise. An incidence matrix $D \in \mathbb{R}^{n\times \ell}$ for $\mathcal{X}$ has a column per edge, where all elements of the column corresponding to edge $(i, j)$ are 0 except for the $i$-th element which is 1 and the $j$-th which is $-1$. Moreover, any incidence matrix for $\mathcal{X}$ satisfies that $\mathbb{1}^T D = 0$. For more details, please refer to (Godsil and Royle, 2001).

### 2.3 Robust Exact Differentiators

In the context of high order sliding modes (HOSM), an $M$-th order exact differentiator is an online algorithm $D_M$ which takes continuous samples of an input signal $f(t) \in \mathbb{R}$ and has outputs $D^{\mu}_M(t)$, which approaches $f^{(\mu)}(t)$ with zero steady-state error after a finite amount of time for any initial conditions, provided that $|f^{(M+1)}(t)| \leq L$.

This type of algorithms were first proposed in (Levant, 2003). Although there are many variations of this algorithm, the structure in which our proposal relies, is given by the following:

$$\begin{aligned}
\dot{z}_0 &= z_1 - k_0 L^{\frac{1}{M+1}} \lceil \sigma \rfloor^{\frac{M}{M+1}} \\
&\cdots \\
\dot{z}_\mu &= z_{\mu+1} - k_\mu L^{\frac{1+\mu}{M+1}} \lceil \sigma \rfloor^{\frac{M-\mu}{M+1}} \\
&\cdots \\
\dot{z}_M &= -k_M L \text{sign}(\sigma)
\end{aligned} \tag{1}$$

where $\sigma(t) = z_0(t) - f(t)$ and the outputs of the differentiator are $D^{\mu}_M(t) = z_\mu(t)$. The values of $k_\mu$ have to be designed such that system (1) is stable with $L = 1$.

Design procedures for the gains $k_\mu$ have been reported such as the one proposed by Cruz-Zavala and Moreno (2019), based on a Lyapunov function criterion.

## 3. PROBLEM STATEMENT

Consider a multi-agent system consisting of $n$ agents. Each agent is capable of communicating with other agents according to a communication topology defined by a connected graph $\mathcal{X}$. Moreover, each agent is capable of executing an algorithm of the form:

$$\dot{x}_i(t) = f_i(x_i(t), p_i(t), u_i(t)), \quad x_i(0) = x^0_i \tag{2}$$

where $x \in \mathbb{R}^m$ is the internal state of the agent computations, where the dimension $m$ will be defined later, $p_i \in \mathbb{R}^{d_i}$ is a vector of received messages from its $d_i$ neighbors and $u_i \in \mathbb{R}$ is its internal time-varying (in general) reference signal. Moreover, $f_i : \mathbb{R}^m \times \mathbb{R}^{d_i} \times \mathbb{R} \to \mathbb{R}^m$ is the evolution rule for the agent. Each agent share a message with its neighbors of the form

$$p_i(t) = h_i(x_i(t), u_i(t)) \tag{3}$$

and has an output

$$y_i(t) = g_i(x_i(t), u_i(t)). \tag{4}$$

Note that the evolution rule (2) evolves in a distributed way, since for each agent, it only uses their internal data and information shared only by its neighbors. The goal of this multi-agent system is stated as follows.

*Problem 1.* Given the agent configuration in (2),(4) and (3), and a set of local signals $u(t) = [u_1(t), \ldots, u_n(t)]^T$; the dynamic average consensus problem consists in designing the functions $f_i(\bullet, \bullet, \bullet), h_i(\bullet, \bullet)$ and $g_i(\bullet, \bullet)$ such that the individual output signals for each agent are able to track

$$\bar{u}(t) = \frac{1}{n}\sum_{i=1}^{n} u_i(t)$$

with zero steady state error.

In the statement of Problem 1, $f_i(\bullet, \bullet, \bullet), h_i(\bullet, \bullet)$ and $g_i(\bullet, \bullet)$ are not meant to correspond to inherent dynamics of the physical agents. However, they represent the dynamics of the consensus algorithm which we are free to design.

*Remark 1.* Note that since the proposal algorithm will be based in the theory of HOSM, then (2) is more properly described by a differential inclusion and hence its solutions will be understood in the sense of Filippov (see Cortes (2008) for more details).

### 3.1 Linear dynamic consensus

The typical solution to this problem, as described in (Kia et al., 2019), is to take advantage of linear control theory. In this context the agent configuration takes the following form:

$$\begin{aligned}
\dot{x}_i &= \sum_{j=1}^{n} a_{ij}(y_i - y_j), \quad x_i(0) = x^0_i \\
y_i &= u_i - x_i
\end{aligned} \tag{5}$$

where $a_{ij}$ are the components of the adjacency matrix of the communication graph $\mathcal{X}$, each agent share their outputs $p_i = y_i$ only, and the agent internal states are scalar, i.e. $m = 1$. Provided that $\sum_{i=1}^{n} x_i(0) = 0$, it can be shown that this approach reaches dynamical consensus of their outputs, tracking the signal $\bar{u}$ but only with a steady state error bounded by a constant $\kappa$ which depends on the bound for the derivatives of the input signals $u_i$ and the algebraic connectivity of the network. Henceforth, doesn't fully solve Problem 1.

## 4. EXACT DYNAMIC CONSENSUS

In order to solve Problem 1, the proposed algorithm takes advantage of the structure of HOSM in the form of robust exact differentiators, and applies them to the dynamic consensus with the following design. Each agent has an internal state $x_i = [x_{i,0}, \dots, x_{i,M}]^T$ where $m = M + 1$. The proposed algorithm takes the following form:

$$\begin{aligned} \dot{x}_{i,0} &= x_{i,1} - k_0 L^{\frac{1}{M+1}} \sum_{j=1}^{n} a_{ij} \lceil y_i - y_j \rfloor^{\frac{M}{M+1}} \\ &\dots \\ \dot{x}_{i,\mu} &= x_{i,\mu+1} - k_\mu L^{\frac{1+\mu}{M+1}} \sum_{j=1}^{n} a_{ij} \lceil y_i - y_j \rfloor^{\frac{M-\mu}{M+1}} \\ &\dots \\ \dot{x}_{i,M} &= -k_M L \sum_{j=1}^{n} a_{ij} \mathrm{sign}(y_i - y_j) \\ y_i(t) &= u_i(t) - x_{i,0}(t) \end{aligned} \tag{6}$$

where $L \geq \max\left\{\left|u_1^{(M+1)}(t)\right|, \dots, \left|u_n^{(M+1)}(t)\right|\right\}$. Moreover, the output space can be trivially augmented as

$$y_{i,\mu}(t) = u_i^{(\mu)}(t) - x_{i,\mu}(t). \tag{7}$$

This means that the outputs $y_{i,\mu}(t)$ are available internally at the agent, but the shared information is still only $p_i(t) = y_i(t)$ or equivalently $p_i(t) = y_{i,0}(t)$.

As well as other existing methods such as the ones described in (Kia et al., 2019), our approach relies in the following assumption.

*Assumption 1.* The initial conditions for (6) are set to be such that for $0 \leq \mu \leq M$,

$$\sum_{i=1}^{n} x_{i,\mu}(0) = 0 \tag{8}$$

*Remark 2.* Note that Assumption 1 is trivially satisfied provided that all agents initialise as $x_{i,\mu}(0) = 0$.

In this work we will show that if there exists gains $k_\mu$ such that (6) reaches a steady state, such state corresponds to consensus towards the average of the reference signals. In order to do that the following lemma is provided, which states that the average of the $\mu$ - th element of the states of all agents remains constant.

*Lemma 3.* Under Assumption 1, the following identity is satisfied for (6), with $0 \leq \mu \leq M$:

$$\sum_{i=1}^{n} x_{i,\mu}(t) = 0, \quad \forall t \geq 0 \tag{9}$$

**Proof.** First, let $X_\mu = [x_{1,\mu}, \dots, x_{n,\mu}]^T$ be a vector containing the $\mu$-th state of all agents. Moreover define $f_\mu : \mathbb{R} \to \mathbb{R}$ as

$$f_\mu(z) = k_\mu L^{\frac{1+\mu}{M+1}} \lceil z \rfloor^{\frac{M-\mu}{M+1}} \tag{10}$$

and $F_\mu : \mathbb{R}^\ell \to \mathbb{R}^\ell$ by

$$F_\mu(z) = [f_\mu(z_1), \dots, f_\mu(z_\ell)]^T.$$

where $\ell$ is the number of edges in $\mathcal{X}$. Using this notation, it can be verified that (6) can be rewritten as

$$\begin{aligned} \dot{X}_\mu &= X_{\mu+1} - DF_\mu(D^T y) \quad , 0 \leq \mu < M \\ \dot{X}_M &= -DF_M(D^T y) \end{aligned} \tag{11}$$

where $y = [y_1, \dots, y_n]^T$ and $D$ is any incidence matrix of the communication graph $\mathcal{X}$. Moreover, note that $s_\mu = \sum_{i=1}^{n} x_{i,\mu}(t) = \mathbb{1}^T X_\mu$. Furthermore,

$$\dot{s}_M = \mathbb{1}^T \dot{X}_M = -\mathbb{1}^T DF_M(D^T y) = 0 \tag{12}$$

Hence, the value of $s_M(t) = s_M(0) = 0$ remains constant $\forall t \geq 0$. By induction, for $0 \leq \mu < M$,

$$\begin{aligned} \dot{s}_\mu &= \mathbb{1}^T \dot{X}_\mu \\ &= \mathbb{1}^T (X_{\mu+1}(t) - DF_\mu(D^T y)) \\ &= \mathbb{1}^T X_{\mu+1}(0) - \mathbb{1}^T DF_\mu(D^T y) = 0 \end{aligned} \tag{13}$$

where Assumption 1 was used. Hence $\mathbb{1}^T s_\mu = 0, 0 \leq \mu \leq M$, $\forall t \geq 0$ which concludes the proof.

The following theorem states that the steady state points for the algorithm (6) are indeed consensus on the average signal $\bar{u}(t)$ and its derivatives.

*Theorem 4.* Under Assumption 1, given that (6) reaches steady state, the steady state consensus values for (6) are characterized by:

$$y_{1,\mu} = y_{2,\mu} = \dots = y_{n,\mu} = \frac{1}{n} \sum_{i=1}^{n} u_i^{(\mu)}(t) \tag{14}$$

for $0 \leq \mu < M$.

**Proof.** Denote with $Y_\mu = [y_{1,\mu}, \dots, y_{n,\mu}]^T$ as a vector containing the outputs of all agents, hence, $y = [y_1, \dots, y_n]^T = Y_0$. From (11), steady state points $y^*$ of (6) are characterized by $D^T y^* = 0$. Henceforth, since $\mathcal{X}$ is connected, $y^* = \alpha(t)\mathbb{1}$ which is consensus. However, by definition $y^* = Y_0^* = u - X_0^*$. Thus, $\alpha(t) = \frac{1}{n}\mathbb{1}^T u(t) - \frac{1}{n}\mathbb{1}^T X_0^* = \bar{u}(t)$ since $\frac{1}{n}\mathbb{1}^T X_0^* = 0$ by Lemma 3.

Moreover, when $D^T y^* = 0$ is achieved, (6) results in the chain of integrators $\dot{x}_{i,\mu}^* = x_{i,\mu+1}^*$, $0 \leq \mu < M$. In this way $\dot{X}_\mu = X_{\mu+1}$. Henceforth,

$$\begin{aligned} Y_1^* &= \dot{u} - X_1^* = \dot{u} - \dot{X}_0^* \\ &= \dot{u} - \frac{d}{dt}(u - Y_0^*) = \dot{Y}_0^* = \dot{\bar{u}}\mathbb{1} \end{aligned} \tag{15}$$

and by induction

$$\begin{aligned} Y_\mu^* &= u^{(\mu)} - X_\mu^* = u^{(\mu)} - \dot{X}_{\mu-1}^* \\ &= u^{(\mu)} - \frac{d}{dt}\left(u^{(\mu-1)} - Y_{\mu-1}^*\right) = \dot{Y}_{\mu-1}^* = \bar{u}^{(\mu)}\mathbb{1} \end{aligned} \tag{16}$$

which concludes the proof.

*Remark 1.* It is important to note that Theorem 4 assumes that (6) reaches steady state. Conditions on the gains $k_\mu$ which guarantee that this is indeed the case will be presented in a later work.

In order to summarize the implications of the protocol (6) and Theorem 4, the following remarks are provided.

*Remark 2.* Note that according to Theorem 4, with an appropriate selection of gains, protocol (6) manages to solve Problem 1 since, through a distributed solution, achieves zero steady-state error, provided that the $(M+1)$-th derivative is bounded.

*Remark 3.* The values of $u_i^{(\mu)}(t)$ can be calculated internally at each agent by means of a differentiator as the one described in Section 2.3, hence the signals $y_{i,\mu}$ in (7) are easily obtained locally.

*Remark 4.* The proposed algorithm manages to track not only $\bar{u}(t)$ but also $\bar{u}^{(\mu)}(t)$ up to a predefined maximum derivative $M$ by only sharing $y_{i,0}$. Henceforth, in practice the value of $M$ may be different for each agent, according to their needs and capabilities. However, sufficient conditions for such scheme requires further research.

*Remark 5.* Although our result is inspired on the differentiator shown in (1), the protocol (6) is not equivalent to it in any way. Protocol (6) inherits some structure from (1) but the main contribution of this new system is that by construction, the steady state behaviour allows its usage as a distributed average consensus algorithm.

*Remark 6.* If each agent is a robot with motion dynamics of relative degree $r$, then in order apply a control law to track a signal such as $\bar{u}(t)$, its derivatives up to $r$ are needed. This can be achieved by using a classical dynamic consensus algorithm and then applying an $r$-th order differentiator to the estimate of $\bar{u}$ and its derivatives. However, this has two main drawbacks. First, a classical dynamical consensus algorithm will obtain an estimate of $\bar{u}$ with a bounded error. Second, this error which is time-varying in general will add additional error to the differentiator estimate of the derivatives. Using our approach, both problems are eliminated.

## 5. SIMULATIONS

*Example 1.* For the purpose of demonstrating the advantages of the proposal, a simulation scenario is described here with the following configuration. There are $n = 5$ agents with topology described by the following adjacency matrix, or equivalently the incidence matrix

$$A = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&1&1\\ 0&1&1&0&0\\ 0&0&1&0&0 \end{bmatrix}, \quad D = \begin{bmatrix} 1&0&0&0\\ 0&0&0&1\\ -1&-1&-1&0\\ 0&0&1&-1\\ 0&1&0&0 \end{bmatrix}$$

and shown in Figure 1. Each agent has internal reference signals

$$\begin{aligned} u_1(t) &= 1.98\sin(0.18t)\\ u_2(t) &= 0.93\sin(1.86t)\\ u_3(t) &= 2.75\sin(0.40t)\\ u_4(t) &= 1.72\sin(1.25t)\\ u_5(t) &= 1.58\sin(1.12t) \end{aligned} \tag{17}$$

and initial conditions $x_{i,\mu}(0) = 0$, $\forall \mu > 0$ and $x_{i,0}(0)$ given by $0.8, 0.15, 0.02, 0.07, -1.07$ respectively. Note that this initial conditions comply with Assumption 1. The gains $k_\mu$ are chosen as $1.87, 1.73, 0.90, 0.22$ for all agents. Moreover, $M = 3$ and $L = 3$.

The individual trajectories for this experiment are shown in Figure 2, as well as the target $\bar{u}(t)$ in red. Note that all agent are able to track not only $\bar{u}(t)$ but also $\dot{\bar{u}}(t), \ddot{\bar{u}}(t)$ and $\bar{u}^{(3)}(t)$.

*Example 2.* Consider the same situation as in Example 1. This scenario was simulated for the linear dynamic consensus algorithm described in Section 3.1 and was compared to the results of Example 1. The error between the outputs of the algorithms for each agent and the value of $\bar{u}$ is shown

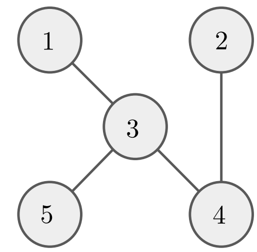


Fig. 1. Graph $\mathcal{X}$ used for simulation Example 1

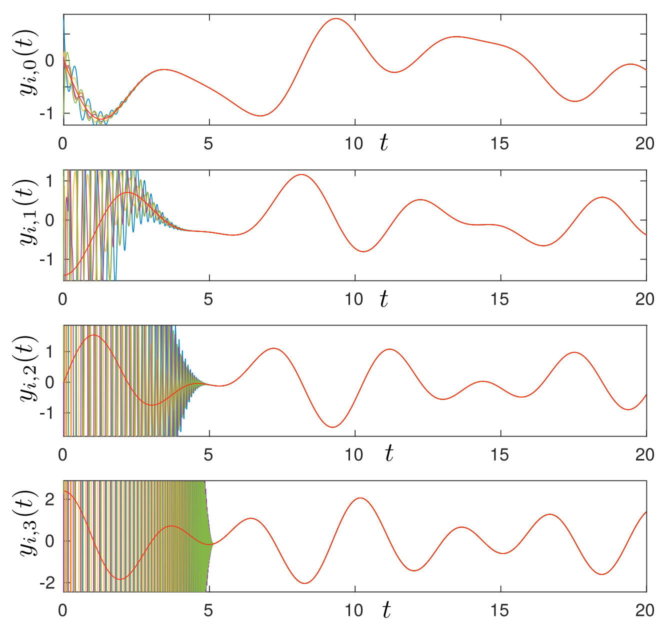


Fig. 2. Resulting trajectories for the agents in the simulation Example 1. The outputs $y_i = y_{i,0}, y_{i,1}$ and $y_{i,2}$ are shown as well as the target signals $\bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}$ and $\bar{u}^{(3)}$ in solid red for reference.

in Figure 3. The individual errors $e_i(t) = |y_i(t) - \bar{u}(t)|$ are shown as a figure of merit. As noted before, our proposal manages to obtain zero-steady state error while the lineal approach only guarantees the error to be bounded by some constant.

*Example 3.* Consider the same scenario as in Example 1. However, in this case, each agent doesn't have access to the reference signals $u_i(t)$ by themselves, but a noisy version of them $\hat{u}_i(t) = u_i(t) + \eta_i(t)$ where $|\eta_i(t)| < \varepsilon = 0.1$. The corresponding trajectories of such scenario for both our proposal and the linear scheme are shown in Figure 4. Tracking does indeed have non-zero steady state error due to the disturbances, achieving only a bounded error for both schemes. Even with this disturbance, the individual tracking from the agents, manage to maintain around $\bar{u}(t)$ within an accuracy band. Nonetheless, this band depends only on the noise for our proposal, and not on the input signals $u_i(t)$, as happens with the linear protocol. Henceforth, it can be observed that the steady state error is much bigger, of up to one order of magnitude, in the linear scheme, when compared to ours.

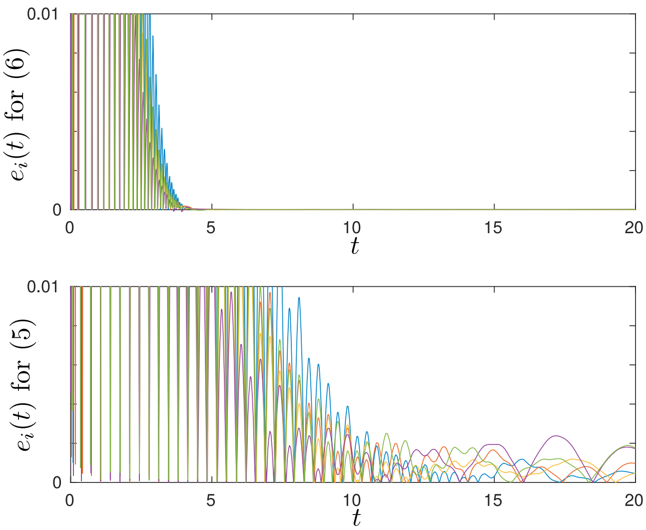


Fig. 3. Comparison of our proposal (top) with a typical implementation of linear dynamical consensus (bottom) for the same simulation scenario as described in Example 2. Individual trajectories of the error $e_i(t) = |y_i(t) - \bar{u}(t)|$ for each agent, are shown as a figure of merit.

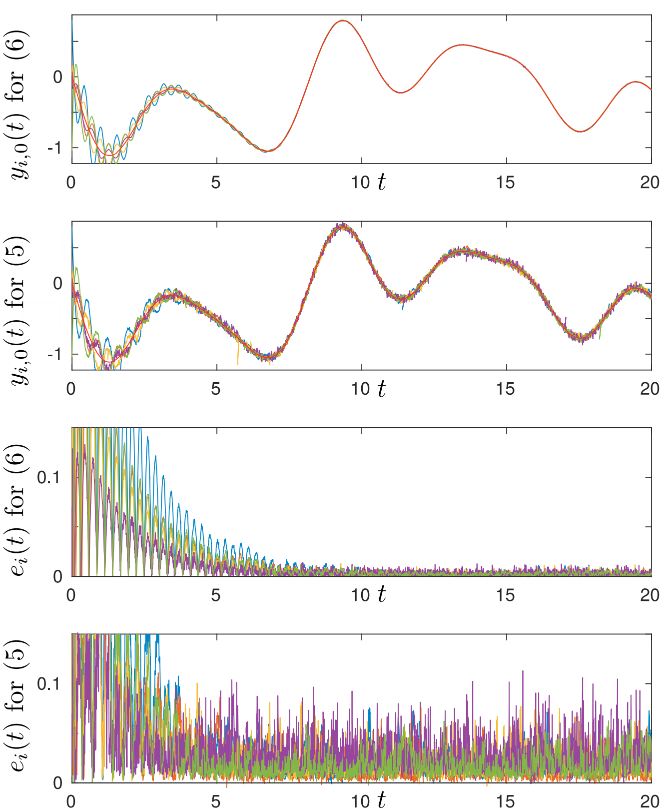


Fig. 4. Simulation results for for the scenario of Example 3. Individual trajectories for each agent, for protocols (6) and (5), as well as $\bar{u}(t)$ in solid red as a reference (top). Individual tracking errors $e_i(t)$ for protocols (6) and (5) (bottom).

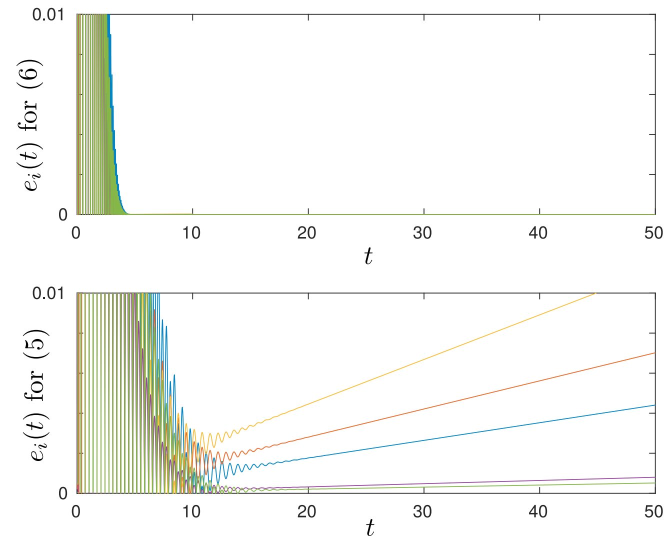


Fig. 5. Comparison of our proposal (top) with a typical implementation of linear dynamical consensus (bottom) for the same simulation scenario as described in Example 4. The error $e_i(t) = |y_i(t) - \bar{u}(t)|$ is shown as a figure of merit.

*Example 4.* This same situation as in Example 1 was simulated for both our approach and the linear dynamic consensus algorithm described in Section 3.1 with the difference that the input signals are now replaced by:

$$\begin{aligned} u_1(t) &= 1.98t^2 \\ u_2(t) &= 0.93t^2 \\ u_3(t) &= 2.75t^2 \\ u_4(t) &= 1.72t^2 \\ u_5(t) &= 1.58t^2 \end{aligned} \tag{18}$$

In this case, the signals $\dot{u}_i(t)$ are unbounded. However, $u^{(\mu)}(t), \mu > 1$ are bounded and the target $\bar{u}(t)$ can be tracked correctly by our proposal. The error between the outputs of the algorithms for each agent and the value of $\bar{u}$ are shown in Figure 5. The individual errors $e_i(t) = |y_i(t) - \bar{u}(t)|$ are shown as a figure of merit. Note that the error for the linear protocol increases linearly since $\dot{u}_i(t)$ increase linearly, without bound. On the other hand, our proposal achieves zero steady-state error.

*Remark 7.* Comparison of high order derivatives in Examples 2,3 and 4 is not possible since the linear approach described in Section 3.1 isn't able to obtain them. However, high gain observers are a linear analogous to the exact linear differentials and can be used similarly as in the approach presented here, in order to achieve practical consensus towards $\bar{u}(t)$ and its derivatives. Nonetheless, this method has the inherited drawback that the steady state error of all the derivative estimates would be bounded by a constant, being zero only if the gain is chosen to be infinite, which is impractical (Vasiljevic and Khalil, 2008). Henceforth, the non-linear approach presented here, is a more scalable and practical solution to the dynamical consensus problem.

## 6. CONCLUSIONS

In this article, an Exact Dynamic Consensus algorithm has been presented, where the agents are able to maintain zero steady-state consensus error, when tracking the average of time-varying signals. The method works under reasonable assumptions about the initial conditions and bounds of certain high order derivatives of the reference signals. The simulation scenario presented here, exposes the effectiveness of our approach in addition to be compared to a classical approach, where the advantage of our proposal is clearly shown. As future work we consider to study concrete stability properties of the proposed algorithm such as finite time convergence. Moreover, analysis of the algorithm under directed communication graphs, switching topologies and discrete communication schemes may be studied.

## REFERENCES

Cortes, J. (2008). Discontinuous dynamical systems. *IEEE Control Systems Magazine*, 28(3), 36–73. doi: 10.1109/MCS.2008.919306.

Cruz-Zavala, E. and Moreno, J.A. (2019). Levant's arbitrary-order exact differentiator: A lyapunov approach. *IEEE Transactions on Automatic Control*, 64(7), 3034–3039.

Godsil, C. and Royle, G. (2001). *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics.* volume 207 of Graduate Texts in Mathematics. Springer.

Gómez-Gutiérrez, D., Vázquez, C.R., Čelikovský, S., Sánchez-Torres, J.D., and Ruiz-León, J. (2018). On finite-time and fixed-time consensus algorithms for dynamic networks switching among disconnected digraphs. *International Journal of Control*, 0(0), 1–15. URL `https://doi.org/10.1080/00207179.2018.1543896`.

Kar, S. and Moura, J.M.F. (2008). Sensor networks with random links: Topology design for distributed consensus. *IEEE Transactions on Signal Processing*, 56(7), 3315–3326. doi:10.1109/TSP.2008.920143.

Kia, S.S., Van Scoy, B., Cortes, J., Freeman, R.A., Lynch, K.M., and Martinez, S. (2019). Tutorial on dynamic average consensus: The problem, its applications, and the algorithms. *IEEE Control Systems Magazine*, 39(3), 40–72. doi:10.1109/MCS.2019.2900783.

Levant, A. (2008). Homogeneous high-order sliding modes. *IFAC Proceedings Volumes*, 41(2), 3799 – 3810. doi:https://doi.org/10.3182/20080706-5-KR-1001.00640. 17th IFAC World Congress.

Levant, A. (2003). Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, 76(9-10), 924–941. doi: 10.1080/0020717031000099029.

Olfati-Saber, R., Fax, J.A., and Murray, R.M. (2007). Consensus and Cooperation in Networked Multi-Agent Systems. *Proceedings of the IEEE*, 95(1), 215–233.

Rahili, S. and Ren, W. (2017). Heterogeneous distributed average tracking using nonsmooth algorithms. In *2017 American Control Conference (ACC)*, 691–696. doi: 10.23919/ACC.2017.7963033.

Vasiljevic, L.K. and Khalil, H.K. (2008). Error bounds in differentiation of noisy signals by high-gain observers. *Systems and Control Letters*, 57(10), 856–862.